Tutorial 10
Advanced Graph Theory
Graph Coloring

27^{th} October 2014

1. You are given finite sets $S_1, ..., S_m$. Let $U = S_1 \times ... \times S_m$. Define a graph G with vertex set U by making $u \leftrightarrow v$ if and only if u and v differ in every coordinate. Determine $\chi(G)$.
2. Now, use the result from Question 1 to answer the following: Consider a traffic signal controlled by two switches, each of which can be set in n positions. For each setting of the switches, the traffic signal shows one of its n possible colors. Whenever the setting of *both* switches changes, the color changes. Prove that the color shown is determined by the position of one of the switches. Interpret this in terms of the chromatic number of some graph. (Greenwell-Lovasz [1974])

3. Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$.

4. **Prove or disprove by counterexample**
   Given an optimal coloring of a k-chromatic graph, for each color i there is a vertex with color i that is adjacent to vertices of the other k-1 colors.

5. **Prove or disprove by counterexample**
   If G is color-critical, then the graph G' generated from it by applying Mycielski's construction is also color-critical.
6. Prove that if G has a proper coloring g in which every color class has atleast two vertices, then G has an optimal coloring f in which every color class has at least two vertices. (Hint: If f has a color class with only one vertex, use g to make an alteration in f. You may also use induction on $\chi(G)$.)
7. Prove that a graph G is 2^k colorable iff G is the union of k bipartite graphs.
8. Every k -chromatic graph has at least ${}^kC_2$ edges.